

Sums of variables at the onset of chaos, replenished

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Abstract

As a counterpart to our previous study of the stationary distribution formed by sums of positions at the Feigenbaum point via the period-doubling cascade in the logistic map (*Eur. Phys. J. B* **87** 32, (2014)), we determine the family of related distributions for the accompanying cascade of chaotic band-splitting points in the same system. By doing this we rationalize how the interplay of regular and chaotic dynamics gives rise to either multiscale or gaussian limit distributions. As demonstrated before (*J. Stat. Mech.* **P01001** (2010)), sums of trajectory positions associated with the chaotic-band attractors of the logistic map lead only to a gaussian limit distribution, but, as we show here, the features of the stationary multiscale distribution at the Feigenbaum point can be observed in the distributions obtained from finite sums with sufficiently small number of terms. The multiscale features are acquired from the repeller preimage structure that dominates the dynamics toward the chaotic attractors. When the number of chaotic bands increases this hierarchical structure with multiscale and discrete scale-invariant properties develops. Also, we suggest that the occurrence of truncated q -gaussian-shaped distributions for specially prescribed sums are t -Student distributions premonitory of the gaussian limit distribution.

1 Introduction

The stationary distributions of sums of consecutive positions of trajectories generated by the logistic map have been found to follow a basic renormalization group (RG) structure: a nontrivial fixed-point multiscale distribution at the period-doubling onset of chaos (Feigenbaum point) and a gaussian trivial fixed-point distribution for all chaotic attractors [1, 2]. At the Feigenbaum point the limit distribution of sums of positions, generated by an ensemble of initial conditions uniformly distributed in the full phase space, possesses an infinite-level hierarchical structure that originates from the properties of the repeller set and its preimages [3]. This ladder organization was elucidated [3] through consideration of the family of periodic attractors, conveniently, the super-stable attractors called supercycles [4], along the period-doubling cascade. Here we complement this study by considering instead the cascade of chaotic band-splitting attractors, or Misiurewicz ($M_n, n = 0, 1, 2, \dots$) points [5]. This gives us the opportunity of analyzing the transformation of the developing multiscale distributions into the gaussian limit distribution. The larger the number 2^n of bands in the attractor the finer the approximation attained to the multiscale structure of the non-trivial fixed-point limit distribution before it undergoes a crossover to the gaussian distribution.

We consider the logistic map $f_\mu(x) = 1 - \mu x^2$, $-1 \leq x \leq 1$, $0 \leq \mu \leq 2$, for which the control parameter value for its main period-doubling cascade accumulation point is $\mu = \mu_\infty = 1.401155189092\dots$ [6]. When μ is shifted to values larger than μ_∞ , $\Delta\mu \equiv \mu - \mu_\infty > 0$, the attractors are (mostly) chaotic and consist of 2^n bands, $n = 0, 1, 2, \dots$, where $2^n \sim \Delta\mu^{-\kappa}$, $\kappa = \ln 2 / \ln \delta$, and $\delta = 4.669201609102\dots$ is the universal constant that measures both the

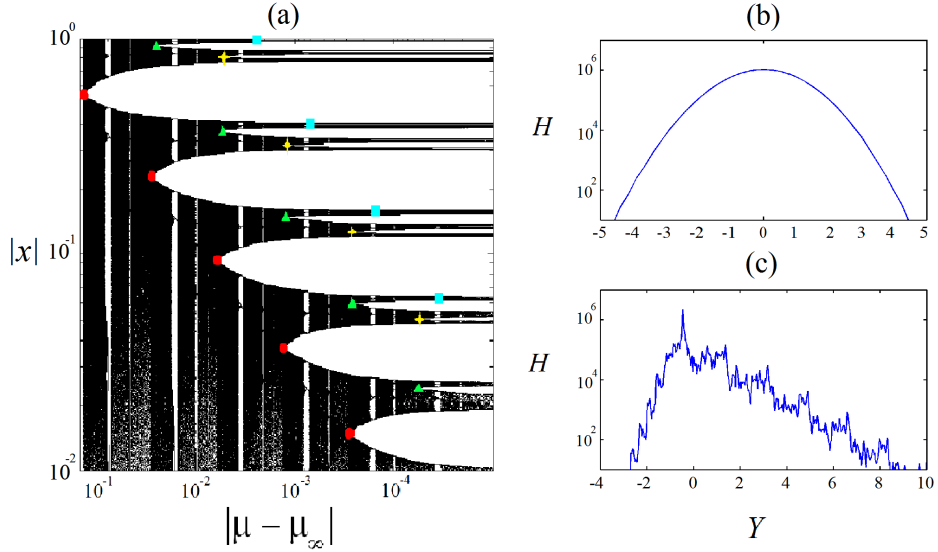


Figure 1: (Colour on-line.) (a) Attractor bands (in black) and gaps between them (white horizontal regions) in logarithmic scales, $-\log(\mu - \mu_\infty)$ and $\log(|x|)$, in the horizontal and vertical axes, respectively. The band-splitting points M_n (circle, triangle, diamond and square symbols) follow straight lines indicative of power-law scaling. The vertical white strips are periodic attractor windows. (b) Histogram H of sums of trajectories when $\mu = 2$ approaches the shape of the gaussian limit distribution. (c) Histogram of sums of trajectories at $\mu = \mu_\infty$ approximates the shape of the multiscale limit distribution.

rate of convergence of the values of $\mu = \mu_n$ to μ_∞ at period doubling or at band splitting points [4]. The latter points are attractor merging crises, where multiple pieces of an attractor merge together at the position of an unstable periodic orbit [7]. The M_n points can be determined by evaluation of the trajectories with initial condition $x_0 = 0$ for different values of μ , as these orbits follow the edges of the chaotic bands until at $\mu = \mu_n$ the unstable orbit of period 2^n reaches the merging crises [7]. In Fig. 1a we show (with different symbols) several M_n points that appear aligned because of the logarithmic scales employed. The family of lines of equal slope along which the M_n points fall reveal the power-law scaling associated with their locations. In Fig. 1b we show a corroboration for the gaussian limit distribution for sums generated at the one-band fully-chaotic attractor at $\mu = 2$, whereas in Fig. 1c we show the multiscale limit distribution for the Feigenbaum point at μ_∞ .

Here we present results for ensembles of sums of positions and their distributions at the attractors where bands split. On the one hand, we find that distributions obtained from sums with an adequately finite number of terms resemble the asymmetric exponential-tailed shape of the limit distribution at μ_∞ , and display gradually more of its multiscale features as the order n of the band-splitting point M_n increases. On the other hand, as the number of summands N increases the distribution for each value of n develops a symmetrical shape and eventually approaches the gaussian limit distribution for $N \rightarrow \infty$, as anticipated for all chaotic attractors. We discuss the occurrence of so-called q -gaussian distributions for special sums of positions at, or around, the M_n points [5], [8], [9] in terms of distributions for finite-size data sets drawn from gaussian variables such as it is the case of the t -Student distribution.

2 Sums of positions and their distributions for band-splitting attractors

We consider the sum of consecutive positions x_t up to a final iteration time N of a trajectory with initial condition x_0 and control parameter value μ , *i.e.*

$$X(x_0, N; \mu) \equiv \sum_{t=0}^N x_t \quad (1)$$

for $x_0 \in [-1, 1]$, as well as the centered, rescaled, sum

$$Y \equiv \frac{X - \langle X \rangle}{\sigma}, \quad (2)$$

where $\langle X \rangle$ is the average value for the sums over the ensemble of \mathcal{M} initial conditions, and σ is the standard deviation of this ensemble. The use of Y instead of X facilitates the comparison between sums with different values of μ_n . We denote by H the histogram corresponding to the sum Y . H can be normalized or not, in which case it will be indicated.

The four panels in Fig. 2 show the results for $X(x_0, N; \mu)$ for all possible initial conditions $-1 \leq x_0 \leq 1$ when the control parameter takes the values for the first four Misiurewicz points, $M_n, n = 0, 1, 2, 3$. The plots are all symmetrical with respect to $x_0 = 0$ and exhibit as a main feature two matching peaks and a central valley. For M_0 the two peaks and the valley are nearly concealed by other symmetrically-located peaks that form the fluctuating background generated by the large chaotic band that forms the attractor. But as the number of bands that form the attractor increases (and their widths decrease) for the subsequent M_1, M_2 and M_3 points the twin large peaks and other, finer, features become increasingly clear. This is the result of weaker intraband chaotic motion fluctuations as the interband regular dynamics progressively dominates.

The panels in Fig. 2 can be compared with those in Fig. 2 in [3] where the corresponding sums for the first supercycle attractors are shown. The same twin peaks separated by a central valley and progressively finer motifs of alternating signs appear in the sums of positions at the supercycle attractors [3], only there the absence of chaotic motion does not mask these details. In [3] it was demonstrated that these features in the sums arise from the dynamics towards the attractors. Namely, the two large peaks arise from the trajectories still close to the main repeller or its first preimage. Whereas the finer peaks of alternating signs are due to trajectory positions near subsequent repellers or their infinite families of preimages. See [3] for details. Here we observe, as shown in Fig. 2 that for the Misiurewicz points the same developments take place and that as the order n of M_n increases their effect on the sums becomes increasingly clear.

We look now at the distributions associated with the sets of sums $X(x_0, N; \mu)$ shown previously. The four panels in Fig. 3 present in semi-logarithmic scales the results for the (normalized) histograms that correspond to the sums in the panels for the first four Misiurewicz points, $M_n, n = 0, 1, 2, 3$ in Fig. 2. In the case of M_0 one obtains an almost round shape with some uneven, pointy, features at the right hand side. In the inset we show the attainment of a gaussian form when the number of summands is increased. The histogram for the next point M_1 already shows a basic tent shape formed by a sharp rise followed by a gentler decline. The following case M_2 displays serrated features over the tent shape that appear even more pronounced in the last case for M_3 . Thus, as n increases we observe the development of the asymmetrical double-exponential global shape, with superimposed motifs of ever-decreasing finer detail, of the stationary distribution for μ_∞ shown in Fig. 1c.

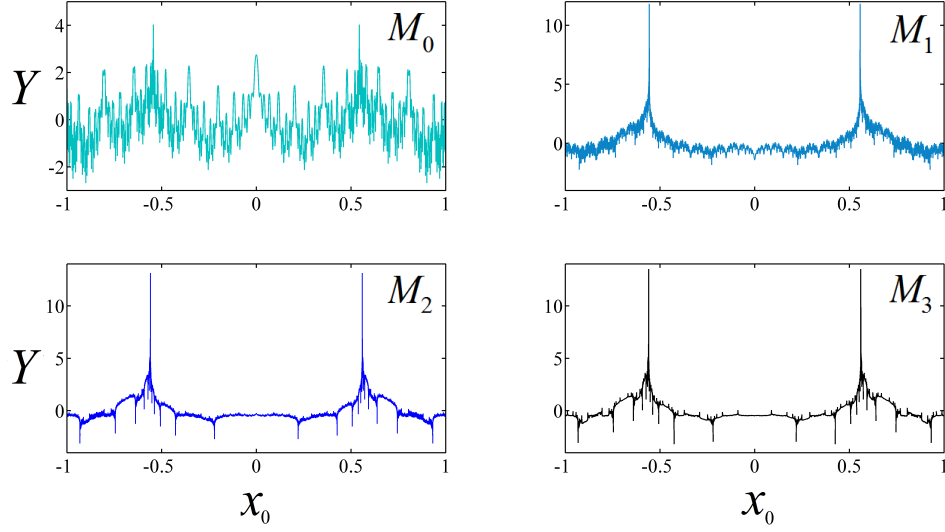


Figure 2: Sums $Y(x_0, N; \mu)$ as a function of the initial condition x_0 . These correspond, as indicated in each frame, to the first four Misiurewicz points, M_0 , M_1 , M_2 and M_3 , when the attractors are about to split into 2, 4, 8 and 16 chaotic bands, respectively. The number of terms in the sums for M_0 , M_1 , M_2 and M_3 are $N = 2^4, 2^5, 2^6, 2^7$, respectively. See text for description.

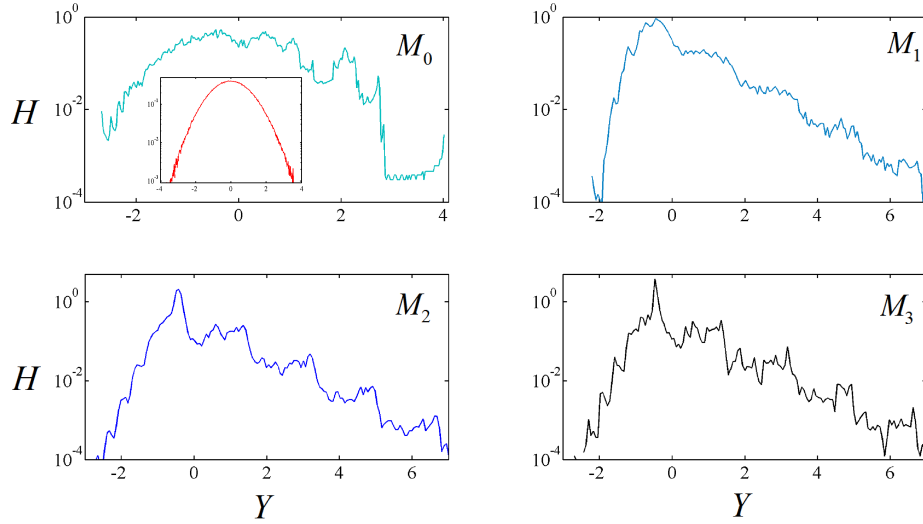


Figure 3: Normalized histograms H , obtained from the sums in Fig. 2. As indicated, these correspond to the first four Misiurewicz points, $M_n, n = 0, 1, 2, 3$. In the inset for M_0 the sum consists of $O(2^{10})$ terms. See text for description

3 Truncated q -gaussian (t -Student) crossover distributions at band-splitting attractors

Special sums of positions of ensembles of trajectories within the chaotic band attractors of the logistic map generate distributions that are well reproduced by truncated q -gaussian shapes [8], [9]. These sums are generated by first discarding an opening set of terms N_s such that the ensemble of trajectories initiated as a uniform distribution within $[-1, 1]$ is already located within the multiple-band attractor [5]. Secondly, the sums are stopped at a total number of terms N_f such that their distribution still displays large non-exponential, but truncated, tails. That is, these kind of sums are of the form

$$X(x_0, N_s, N_f; \mu) \equiv \sum_{t=N_s}^{N_s+N_f} x_t. \quad (3)$$

In Fig. 4 where we show in panels (a), (b) and (c) the centered sums $Y \equiv (X - \langle X \rangle)/\sigma$ for the Misiurewicz points M_3 , M_4 and M_5 , when the attractors are about to split into 16, 32 and 64 chaotic bands, respectively. In panel (d) we show the distributions P for these sums without rescaling of the horizontal axis Y . This kind of distributions have been fitted by q -gaussians [8], [9].

The Student's distribution or t -distribution is defined through the random variable

$$\vartheta = \frac{W\sqrt{n}}{\sqrt{Z}}, \quad (4)$$

where n is the sample size, the random variable X follows a gaussian distribution and Z is a random variable $Z = Z_1^2 + Z_2^2 + Z_3^2 + \dots + Z_n^2$ following a χ^2 -distribution. The sum in Eq. (3) when it is centered and rescaled as in Eq. (2), can be viewed as a random variable similar to that in Eq. (4) by including the crucial and delicate, but rather well-justified, assumption that the sums of positions follow a gaussian distribution. This is possible provided that the correlation function for the positions x_t goes to zero as the number of iterations N goes to infinity in a chaotic regime, as it is certainly fulfilled by the collection of Misiurewicz points we have used [6]. Under this assumption, the quantity in the numerator $\tilde{X} \equiv X - \langle X \rangle$ will (asymptotically) follow a gaussian distribution, and, considering the fact that the standard deviation in Eq. (2) is $\sigma = (1/\mathcal{M} \sum_i^{\mathcal{M}} (X_i - \langle X \rangle))^2$ we can write

$$Y = \frac{\tilde{X}\sqrt{\mathcal{M}}}{\sqrt{\tilde{Z}}}, \quad (5)$$

where we have considered \tilde{Z} to be $\tilde{Z} = \sum_i^{\mathcal{M}} \tilde{X}_i^2$. The probability density function of the variable defined in Eq. (4) is

$$P(\vartheta) = \frac{\Gamma(\frac{n+1}{2})}{(n\pi)^{1/2}\Gamma(\frac{n}{2})} \left(1 + \frac{\vartheta^2}{n}\right)^{-\frac{n+1}{2}}. \quad (6)$$

So we can expect, and then corroborate numerically, that our variable Y follows the same distribution, hence the sums of iterates Y can be regarded to follow a t -distribution. It is well known that the t -distribution arises when estimating the mean of finite data sets drawn from an infinite set that pertains to a gaussian distribution [11]. The larger the sample the closer the t -distribution resembles the gaussian distribution and, in the limit $n \rightarrow \infty$ this is no longer an approximation and we recover a Gaussian, as we can check straightforwardly from Eq. (6).

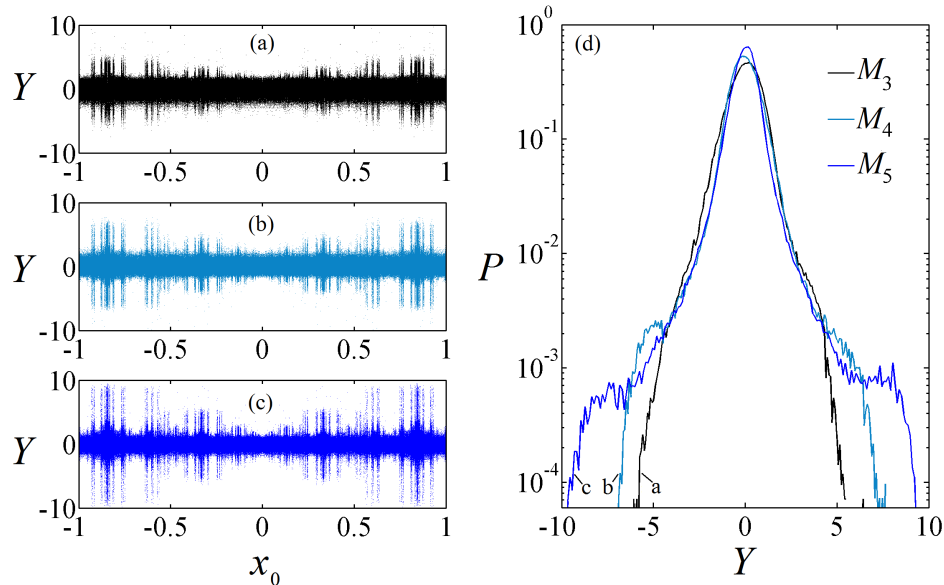


Figure 4: (Colour on-line.) Rescaled sums obtained from a uniform distribution of 10^6 initial conditions across $[-1, 1]$ at the band-splitting points M_n , $n = 3, 4, 5$ with labels (a),(b),(c), respectively, and with their corresponding distributions (normalized histograms) P in (d). The values of N_s and N_f used are, respectively $N_s = 2^6, 2^7, 2^8$, and $N_f = 2^9, 2^{10}, 2^{11}$.

It has been demonstrated [12] that this distribution optimizes the Tsallis entropy expression [13]. The expression for the t -distribution coincides (after a suitable change of variable) with that of the so-called q -gaussian distribution [13]

$$P(y) = \left(\frac{\beta(q-1)}{\pi} \right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)} (1 + \beta(q-1)y^2)^{\frac{1}{1-q}}, \quad 1 < q < 3. \quad (7)$$

Therefore, the appearance of q -gaussian-like distributions for these families of sums are predictable as the chosen finite sets of data cannot still approximate the gaussian limit distribution. Interestingly, the t -Student (or q -gaussian) distribution maximizes an entropy expression that satisfies only three of the Khinchin axioms (with the exclusion of composability) [14] whereas the gaussian distribution maximizes the canonical entropy expression that satisfies all four Khinchin axioms.

4 Summary and discussion

We have probed the dynamical properties of chaotic attractors of the logistic map with the intention of learning about the distributions formed by sums of ensembles of positions of trajectories with uniformly-distributed initial conditions through the total phase space $[-1, 1]$. For suitability we chose members of the family of attractors at which bands split. Our results are consistent with the existence of a single type of limit distribution (i.e. when the number of summands $N \rightarrow \infty$) for all chaotic attractors, this being the gaussian distribution. But we also observed, when the sums have a sufficiently small number of terms, the characteristic features of the multiscale, asymmetric, exponential-tailed, limit distribution that occurs for the special case of the accumulation point of the band-splitting attractors, the Feigenbaum point at μ_∞ . The crossover that takes place from distributions displaying the multiscale form

to the gaussian shape was also addressed. Sums obtained when all trajectories are placed within the chaotic bands and are built-up from finite sets of positions that evolve randomly are comparable to sets of variables extracted from a gaussian population as this is their limit distribution. Such procedure generates the t -Student distribution, equivalently a q -gaussian distribution, as previous studies detected.

Acknowledgments

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